

Co-positivity of tensors and Stability conditions of CP conserving two-Higgs-doublet potential*

Yisheng Song[†]

Abstract

We give how to calculate the vacuum stability conditions of scalar potential for two Higgs doublet model with explicit CP conservation. Moreover, the analytical sufficient and necessary conditions are obtained. The argument methods are first used to study the tensor with two parameters.

Keywords: Co-positivity, Fourth order tensors, Homogeneous polynomial, 2HDM

1 Introduction

The stability model of multi-Higgs potential is very noticeable in particle physics community, and such a model itself was first proposed by Lee [1] for the two-Higgs-doublet model (for short, 2HDM) in 1973. Weinberg [2] gave a general model of multi-Higgs potential in 1976. It was studied in hundreds of papers since then, one of the simplest extensions of the standard model is the 2HDM. There are many studies on the bounded from below (for short, BFB) or the vacuum stability conditions of the 2HDM potential, including CP conservation and CP violation. These results are different kinds of analytical conditions of the vacuum stability of such a potential, for examples, 2HDM with CP conservation in Refs. [3–9], the most general 2HDM in Refs. [3, 6], 2HDM with CP conservation and CP violation in Ref. [5, 6, 10, 11], 2HDM handled numerically [12] and others references that are no cited here. For the tree-level metastability bounds of the most general 2HDM see Ref. [13]. Recently, H. Bahl et al. [14] presented an

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[†]Corresponding author. School of Mathematical Sciences, Chongqing Normal University, Chongqing 401331 P.R. China, Email: yisheng.song@cqn.edu.cn

analytically sufficient condition of the vacuum stability of 2HDM with CP violation. However, there are not a simple analytical expression in the vacuum stability conditions even for 2HDM until now. In this paper, we will try our best to give a argument technique to solve this problem, and provide a simple analytical expression (Theorem 1) of the vacuum stability for 2HDM with CP conservation.

It is well-known that for the 2HDM with explicit CP conservation, all couplings of the Higgs potential are real [1, 3, 10, 13]. The scalar potential of such a 2HDM with Higgs doublets Φ_1 and Φ_2 is

$$\begin{aligned} V_H(\Phi_1, \Phi_2) &= V_2(\Phi_1, \Phi_2) + V_4(\Phi_1, \Phi_2) \\ V_2(\Phi_1, \Phi_2) &= m_{11}^2 |\Phi_1|^2 + m_{22}^2 |\Phi_2|^2 - m_{12}^2 (\Phi_1^* \Phi_2 + \Phi_2^* \Phi_1) \\ V_4(\Phi_1, \Phi_2) &= \Lambda_1 |\Phi_1|^4 + \Lambda_2 |\Phi_2|^4 + \Lambda_3 |\Phi_1|^2 |\Phi_2|^2 \\ &\quad + \Lambda_4 (\Phi_1^* \Phi_2)(\Phi_2^* \Phi_1) + \frac{\Lambda_5}{2} [(\Phi_1^* \Phi_2)^2 + (\Phi_2^* \Phi_1)^2] \\ &\quad + \Lambda_6 |\Phi_1|^2 (\Phi_1^* \Phi_2 + \Phi_2^* \Phi_1) \\ &\quad + \Lambda_7 |\Phi_2|^2 (\Phi_1^* \Phi_2 + \Phi_2^* \Phi_1), \end{aligned} \tag{1}$$

where Φ^* is Hermitian conjugate of Φ . The stability (the BFB) of the scalar potential in the 2HDM is considered only the non-negativity of the quartic part V_4 [10], i.e., $V_4(\Phi_1, \Phi_2) \geq 0$.

In this paper, with the help of the corresponding theory and methods of higher order tensors, we mainly present the sufficient and necessary conditions of the vacuum stability for the 2HDM potential with explicit CP conservation. That is, our main result is the following:

Theorem 1. *Let $\Lambda_1 > 0$, $\Lambda_2 > 0$. Then $V_4(\Phi_1, \Phi_2) \geq 0$ if and only if*

- (1) $\Lambda_6 = \Lambda_7 = 0$, $\Lambda_3 + 2\sqrt{\Lambda_1 \Lambda_2} \geq 0$, $\Lambda_3 + \Lambda_4 - |\Lambda_5| + 2\sqrt{\Lambda_1 \Lambda_2} \geq 0$;
- (2) $\Delta \geq 0$, $\Lambda_3 + 2\sqrt{\Lambda_1 \Lambda_2} \geq 0$,

$$|\Lambda_6 \sqrt{\Lambda_2} - \Lambda_7 \sqrt{\Lambda_1}| \leq 2\sqrt{\Lambda_1 \Lambda_2 (\Lambda_3 + \Lambda_4 + \Lambda_5)} + 2\Lambda_1 \Lambda_2 \sqrt{\Lambda_1 \Lambda_2},$$

$$(i) -2\sqrt{\Lambda_1 \Lambda_2} \leq \Lambda_3 + \Lambda_4 + \Lambda_5 \leq 6\sqrt{\Lambda_1 \Lambda_2},$$

$$(ii) \Lambda_3 + \Lambda_4 + \Lambda_5 > 6\sqrt{\Lambda_1 \Lambda_2} \text{ and}$$

$$|\Lambda_6 \sqrt{\Lambda_2} + \Lambda_7 \sqrt{\Lambda_1}| \leq 2\sqrt{\Lambda_1 \Lambda_2 (\Lambda_3 + \Lambda_4 + \Lambda_5)} - 2\Lambda_1 \Lambda_2 \sqrt{\Lambda_1 \Lambda_2},$$

where $\Delta = 4(12\Lambda_1 \Lambda_2 - 12\Lambda_6 \Lambda_7 + (\Lambda_3 + \Lambda_4 + \Lambda_5)^2)^3 - (72\Lambda_1 \Lambda_2 (\Lambda_3 + \Lambda_4 + \Lambda_5) + 36\Lambda_6 \Lambda_7 (\Lambda_3 + \Lambda_4 + \Lambda_5) - 2(\Lambda_3 + \Lambda_4 + \Lambda_5)^3 - 108\Lambda_1 \Lambda_7^2 - 108\Lambda_6^2 \Lambda_2)^2$.

2 2HDM potential and Real tensors

In order to show our main result, we need turn the polynomial $V_4(\Phi_1, \Phi_2)$ about two complex variable into a 4th order symmetric real tensor, and

then, use some conclusions of tensors to prove our conclusion.

Let $\phi_k = |\Phi_k|$, the modulus of Φ_k for $k = 1, 2$. Then

$$\Phi_1^* \Phi_2 = \phi_1 \phi_2 \rho e^{i\theta} \text{ and } \Phi_2^* \Phi_1 = \phi_1 \phi_2 \rho e^{-i\theta},$$

here $i^2 = -1$ and $\rho \in [0, 1]$ is the orbit space parameter [7, 11]. So, we have

$$\begin{aligned} V_4(\Phi_1, \Phi_2) &= \Lambda_1 \phi_1^4 + \Lambda_2 \phi_2^4 + \Lambda_3 \phi_1^2 \phi_2^2 + \Lambda_4 \rho^2 \phi_1^2 \phi_2^2 \\ &\quad + \frac{\Lambda_5}{2} (\phi_1^2 \phi_2^2 \rho^2 e^{2i\theta} + \phi_1^2 \phi_2^2 \rho^2 e^{-2i\theta}) \\ &\quad + \Lambda_6 \phi_1^2 (\phi_1 \phi_2 \rho e^{i\theta} + \phi_1 \phi_2 \rho e^{-i\theta}) \\ &\quad + \Lambda_7 \phi_2^2 (\phi_1 \phi_2 \rho e^{i\theta} + \phi_1 \phi_2 \rho e^{-i\theta}) \\ &= \Lambda_1 \phi_1^4 + \Lambda_2 \phi_2^4 + \Lambda_3 \phi_1^2 \phi_2^2 + \Lambda_4 \rho^2 \phi_1^2 \phi_2^2 \\ &\quad + \Lambda_5 \rho^2 \phi_1^2 \phi_2^2 \cos 2\theta + 2\Lambda_6 \phi_1^3 \phi_2 \rho \cos \theta + 2\Lambda_7 \phi_2^3 \phi_1 \rho \cos \theta \\ &= \Lambda_1 \phi_1^4 + \Lambda_2 \phi_2^4 + (\Lambda_3 + \Lambda_4 \rho^2 - \Lambda_5 \rho^2) \phi_1^2 \phi_2^2 \\ &\quad + 2\Lambda_5 \rho^2 \phi_1^2 \phi_2^2 \cos^2 \theta + 2\Lambda_6 \rho \phi_1^3 \phi_2 \cos \theta + 2\Lambda_7 \rho \phi_2^3 \phi_1 \cos \theta. \end{aligned}$$

Let $x = \cos \theta$. Then $x \in [-1, 1]$ and

$$\begin{aligned} V_4(\Phi_1, \Phi_2) &= \Lambda_1 \phi_1^4 + \Lambda_2 \phi_2^4 + (\Lambda_3 + \Lambda_4 \rho^2 - \Lambda_5 \rho^2) \phi_1^2 \phi_2^2 \\ &\quad + 2\Lambda_5 \rho^2 \phi_1^2 \phi_2^2 x^2 + 2\rho(\Lambda_6 \phi_1^2 + \Lambda_7 \phi_2^2) \phi_1 \phi_2 x. \end{aligned} \tag{2}$$

This defines a 4th-order 2-dimensional symmetric tensor $\mathcal{A}(\rho, x) = (a_{ijkl})$ with a parameter ρ and x :

$$\begin{aligned} a_{1111} &= \Lambda_1, a_{2222} = \Lambda_2, \\ a_{1122} &= \frac{1}{6} [\Lambda_3 + \Lambda_4 \rho^2 + \Lambda_5 \rho^2 (2x^2 - 1)], \\ a_{1112} &= \frac{1}{2} \Lambda_6 \rho x, \quad a_{1222} = \frac{1}{2} \Lambda_7 \rho x. \end{aligned} \tag{3}$$

So the vacuum stability (or the BFB) of the system $V_H(\Phi_1, \Phi_2)$ may turn into the co-positivity of a 4th-order tensor $\mathcal{A}(\rho, x)$.

The positive definiteness and the co-positivity of a 4th order symmetric tensor are applied to deal with the vacuum stability conditions of the particle physics model in Ref. [7]. Recently, In Refs. [15–19], the distinctly sufficient conditions were given for the co-positivity of 4th order 3-dimensional symmetric tensors, which may be used to the vacuum stability conditions of scalar potential of the particle physics model.

3 Co-positivity of Matrices and Tensors

The co-positivity of a matrix $M = (m_{ij})$ is used to verify the vacuum stability of the particle physics model in Refs. [7, 8]. A symmetric

matrix $M = (m_{ij})$ is co-positive if the quadratic form $x^\top Mx \geq 0$ for all non-negative vectors $x \in \mathbb{R}^n$. The co-positivity of a 2×2 symmetric matrix $M = (m_{ij})$ was showed in Ref. [20, Lemma 2.1], also see Hadeler [21, Theorem 2] and Nadler [22, Lemma 1] for more details. A 2×2 symmetric matrix $M = (m_{ij})$ is co-positive if and only if

$$m_{11} \geq 0, m_{22} \geq 0 \text{ and } m_{12} + \sqrt{m_{11}m_{22}} \geq 0.$$

The co-positivity of a symmetric tensor is tried to test the vacuum stability of the particle physical model in Ref. [7]. A m th-order n -dimensional symmetric tensor $T = (t_{i_1 \dots i_m})(i_j = 1, 2, \dots, n, j = 1, 2, \dots, m)$ is co-positive [24–28] if the quartic form $T\mathbf{x}^m \geq 0$ for all non-negative vectors $\mathbf{x} \in \mathbb{R}^n$, where $\mathbf{x} = (x_1, x_2, \dots, x_n)^\top$ and

$$T\mathbf{x}^m = \mathbf{x}^\top (T\mathbf{x}^{m-1}) = \sum_{i_1 \dots i_m=1}^n t_{i_1 \dots i_m} x_{i_1} \cdots x_{i_m},$$

$T\mathbf{x}^{m-1} = (y_1, y_2, \dots, y_n)^\top$ is a vector with its entiries

$$y_k = (T\mathbf{x}^{m-1})_k = \sum_{i_2 \dots i_m=1}^n t_{k i_2 \dots i_m} x_{i_2} \cdots x_{i_m}.$$

Let $f(x, y)$ be a quartic homogeneous real polynomial about two variables x, y ,

$$f(x, y) = a_0x^4 + a_1x^3y + a_2x^2y^2 + a_3xy^3 + a_4y^4. \quad (4)$$

Then it gives a 4th-order 2-dimensional symmetric tensor $T = (t_{ijkl})$ with its entires,

$$t_{1111} = a_0, t_{2222} = a_4, t_{1122} = \frac{1}{6}a_2, t_{1112} = \frac{1}{4}a_1, t_{1222} = \frac{1}{4}a_3,$$

Assume that $a_0 > 0$ and $a_4 > 0$. In Ref. [16], the co-positivity of the above tensor T was proved (also see Refs. [29, 30]).

Lemma 1. [16, Lemma 3.1] Let $a_0 > 0$ and $a_4 > 0$. Then $f(x, y) \geq 0$ for all $x \geq 0, y \geq 0$ if and only if

- (1) $\Delta \leq 0$ and $a_1\sqrt{a_4} + a_3\sqrt{a_0} > 0$; or
- (2) $a_1 \geq 0, a_3 \geq 0$ and $2\sqrt{a_0a_4} + a_2 \geq 0$; or
- (3) $\Delta \geq 0, |a_1\sqrt{a_4} - a_3\sqrt{a_0}| \leq 4\sqrt{a_0a_2a_4 + 2a_0a_4\sqrt{a_0a_4}}$ and
 - (i) $-2\sqrt{a_0a_4} \leq a_2 \leq 6\sqrt{a_0a_4}$,
 - (ii) $a_2 > 6\sqrt{a_0a_4}$ and

$$a_1\sqrt{a_4} + a_3\sqrt{a_0} \geq -4\sqrt{a_0a_2a_4 - 2a_0a_4\sqrt{a_0a_4}},$$

where $\Delta = 4(12a_0a_4 - 3a_1a_3 + a_2^2)^3 - (72a_0a_2a_4 + 9a_1a_2a_3 - 2a_2^3 - 27a_0a_3^2 - 27a_1^2a_4)^2$.

4 Vacuum stability conditions

In this section, we mainly give the vacuum stability conditions of the 2HDM with explicit CP conservation. That is, how to logically establish our main result, Theorem 1.

The quartic part (2) of such a CP conserving two-Higgs-doublet potential may be rewritten as follow

$$\begin{aligned} V_4(\Phi_1, \Phi_2) = & 2\Lambda_5\rho^2\phi_1^2\phi_2^2x^2 + 2\rho(\Lambda_6\phi_1^2 + \Lambda_7\phi_2^2)\phi_1\phi_2x \\ & + \Lambda_1\phi_1^4 + \Lambda_2\phi_2^4 + (\Lambda_3 + \Lambda_4\rho^2 - \Lambda_5\rho^2)\phi_1^2\phi_2^2. \end{aligned}$$

If $\phi_1 > 0$, $\phi_2 = 0$ (or $\phi_1 = 0$, $\phi_2 > 0$), then $V_H(\Phi_1, \Phi_2) = \Lambda_1\phi_1^4$ (or $\Lambda_2\phi_2^4$). Without loss of generality, we may assume $\Lambda_1 > 0$, $\Lambda_2 > 0$, $\phi_1 > 0$, $\phi_2 > 0$ in the sequel. Let

$$A = 2\Lambda_5\rho^2\phi_1^2\phi_2^2, \quad B = 2\rho(\Lambda_6\phi_1^2 + \Lambda_7\phi_2^2)\phi_1\phi_2, \quad (5)$$

$$C = \Lambda_1\phi_1^4 + \Lambda_2\phi_2^4 + [\Lambda_3 + (\Lambda_4 - \Lambda_5)\rho^2]\phi_1^2\phi_2^2. \quad (6)$$

Then, $V_4(\Phi_1, \Phi_2)$ may be seen as a quadratic function $f(x)$ about a variable $x \in [-1, 1]$,

$$f(x) = V_H(\Phi_1, \Phi_2) = Ax^2 + Bx + C. \quad (7)$$

So, if $A > 0$, then as you can see from the image below, the function $f(x)$ attains its minimum $\frac{4AC-B^2}{4A}$ at a point $x = -\frac{B}{2A} \in [-1, 1]$ or attains its minimum $A + C \pm B$ at a point $x = \pm 1$ ($-\frac{B}{2A} \notin [-1, 1]$). At that time, the graph-like of $f(x)$ is as shown below:

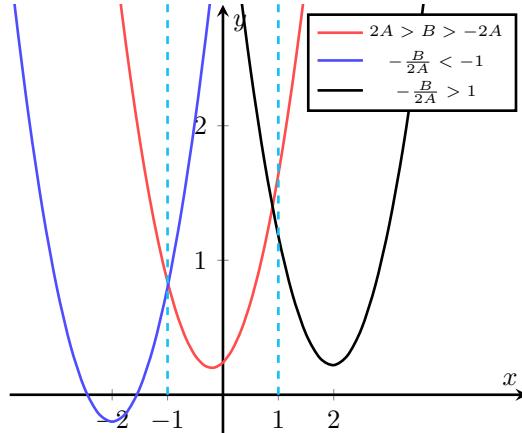
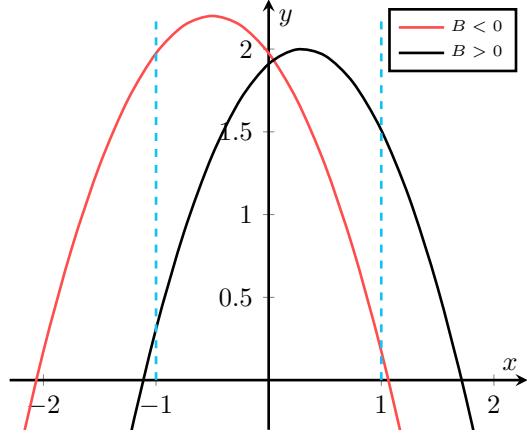


Figure 1: Graph of $f(x)$ ($A > 0$)

If $A \leq 0$, then $f(x)$ attains its minimum $A - B + C$ or $A + B + C$ at a point $x = -1$ ($B \geq 0$) or $x = 1$ ($B < 0$). At that time, the graph-like of $f(x)$ is as shown below:

Figure 2: Graph of $f(x)$ ($A < 0$)

So the following conclusions is easy to be obtained. For detail proof, see Appendix.

Proposition 2. $V_H(\Phi_1, \Phi_2) \geq 0$ if and only if

$$\begin{cases} 4AC - B^2 \geq 0, -2A \leq B \leq 2A, \\ A - B + C \geq 0, \\ A + B + C \geq 0. \end{cases}$$

Proposition 3. $4AC - B^2 \geq 0$ and $-2A \leq B \leq 2A$ for all $\phi_1 \geq 0, \phi_2 \geq 0$ if and only if $\Lambda_6 = 0, \Lambda_7 = 0, \Lambda_5 \geq 0$,

$$\Lambda_3 + \Lambda_4 - \Lambda_5 + 2\sqrt{\Lambda_1 \Lambda_2} \geq 0, \quad \Lambda_3 + 2\sqrt{\Lambda_1 \Lambda_2} \geq 0.$$

It follows from the equations (5) and (6) that

$$\begin{aligned} A \pm B + C &= 2\Lambda_5 \rho^2 \phi_1^2 \phi_2^2 \pm 2\rho(\Lambda_6 \phi_1^2 + \Lambda_7 \phi_2^2) \phi_1 \phi_2 \\ &\quad + \Lambda_1 \phi_1^4 + \Lambda_2 \phi_2^4 + [\Lambda_3 + (\Lambda_4 - \Lambda_5)\rho^2] \phi_1^2 \phi_2^2 \\ &= (\Lambda_4 + \Lambda_5) \phi_1^2 \phi_2^2 \rho^2 \pm 2(\Lambda_6 \phi_1^2 + \Lambda_7 \phi_2^2) \phi_1 \phi_2 \rho \\ &\quad + \Lambda_1 \phi_1^4 + \Lambda_2 \phi_2^4 + \Lambda_3 \phi_1^2 \phi_2^2 \\ &= a\rho^2 \pm b\rho + c, \end{aligned}$$

where

$$a = (\Lambda_4 + \Lambda_5) \phi_1^2 \phi_2^2, \quad b = 2(\Lambda_6 \phi_1^2 + \Lambda_7 \phi_2^2) \phi_1 \phi_2, \quad (8)$$

$$c = \Lambda_1 \phi_1^4 + \Lambda_2 \phi_2^4 + \Lambda_3 \phi_1^2 \phi_2^2. \quad (9)$$

For $\rho \in [0, 1]$, Let

$$s(\rho) = a\rho^2 + b\rho + c \text{ and } t(\rho) = a\rho^2 - b\rho + c. \quad (10)$$

So, if $a > 0$, then the function $s(\rho)$ reaches its minimum $\frac{4ac-b^2}{4a}$ at a point $\rho = -\frac{b}{2a} \in [0, 1]$, or attains its minimum c (or $a + c + b$) at a point $\rho = 0$ (or 1) ($-\frac{b}{2a} \notin [0, 1]$). At that moment, the graph-like of quadratic function $s(\rho)$ is illustrated by the following image:

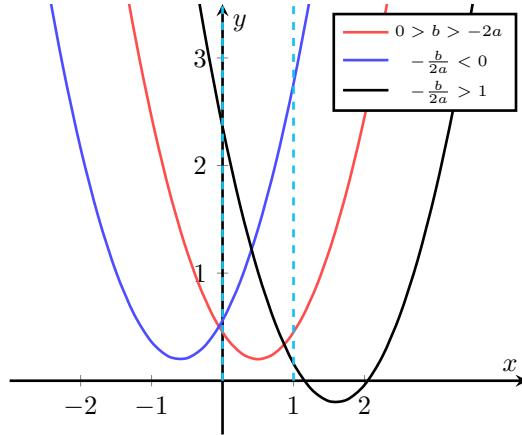


Figure 3: Graph of $s(\rho)$ ($a > 0$)

If $a \leq 0$, then $s(\rho)$ attains its minimum c or $a + b + c$ at a point $\rho = 0$ ($-\frac{b}{2a} \geq \frac{1}{2}$) or $\rho = 1$ ($-\frac{b}{2a} \leq \frac{1}{2}$); if $a = 0$, then $s(\rho)$ attains its minimum c or $b + c$ at a point $\rho = 0$ ($b \geq 0$) or $\rho = 1$ ($b \leq 0$). The graph-like of $s(\rho)$ is as shown below:

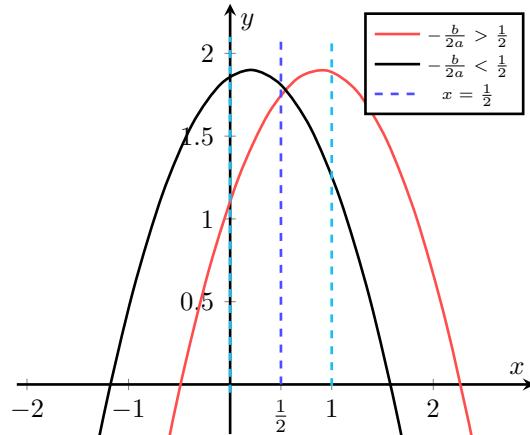
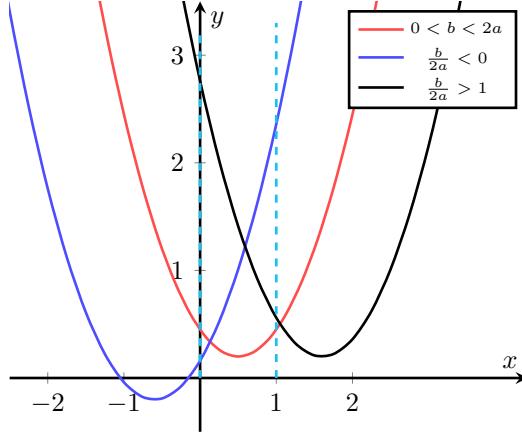
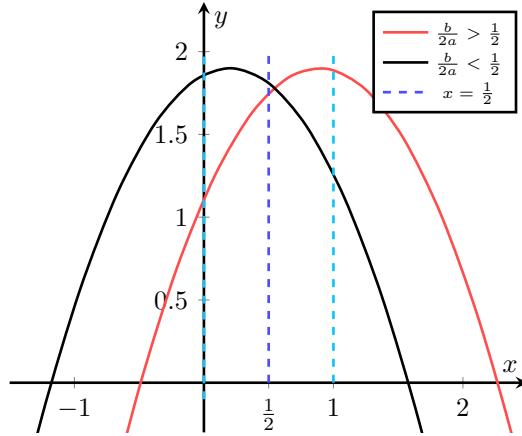


Figure 4: Graph of $s(\rho)$ ($a < 0$)

Similarly, if $a > 0$, then the function $t(\rho)$ reaches its minimum $\frac{4ac-b^2}{4a}$ at a point $\frac{b}{2a} \in [0, 1]$, or attains its minimum c (or $a + c + b$) at a point $\rho = 0$ (or 1) ($\frac{b}{2a} \notin [0, 1]$). The graph-like of $t(\rho)$ is the image below:

Figure 5: Graph of $t(\rho)$ ($a > 0$)

if $a < 0$, then $t(\rho)$ attains its minimum c or $a - b + c$ at a point $\rho = 0$ ($\frac{b}{2a} \geq \frac{1}{2}$) or $\rho = 1$ ($\frac{b}{2a} \leq \frac{1}{2}$); if $a = 0$, then $t(\rho)$ attains its minimum c or $-b + c$ at a point $\rho = 0$ ($b \leq 0$) or $\rho = 1$ ($b \geq 0$). The graph-like of $t(\rho)$ is the following:

Figure 6: Graph of $t(\rho)$ ($a < 0$)

So the following conclusions is easy to be obtained. For detail proof, see Appendix.

Proposition 4. (1) $A + B + C \geq 0$ if and only if

$$\begin{cases} \frac{4ac-b^2}{4a} \geq 0, -2a \leq b \leq 0; \\ c \geq 0; \\ a + b + c \geq 0. \end{cases}$$

(2) $A - B + C \geq 0$ if and only if

$$\begin{cases} \frac{4ac-b^2}{4a} \geq 0, 0 \leq b \leq 2a; \\ c \geq 0; \\ a - b + c \geq 0. \end{cases}$$

Proposition 5. (1) $c \geq 0$ if and only if

$$\Lambda_1 \geq 0, \Lambda_2 \geq 0, \Lambda_3 + 2\sqrt{\Lambda_1 \Lambda_2} \geq 0.$$

(2) $\frac{4ac-b^2}{4a} \geq 0$ and $-2a \leq b \leq 0$ if and only if $\Lambda_4 + \Lambda_5 \geq 0$,

$$\Lambda_6 = 0, \Lambda_7 = 0, \Lambda_1 \geq 0, \Lambda_2 \geq 0, \Lambda_3 + 2\sqrt{\Lambda_1 \Lambda_2} \geq 0.$$

(3) $\frac{4ac-b^2}{4a} \geq 0$ and $2a \geq b \geq 0$ if and only if $\Lambda_4 + \Lambda_5 \geq 0$,

$$\Lambda_6 = 0, \Lambda_7 = 0, \Lambda_1 \geq 0, \Lambda_2 \geq 0, \Lambda_3 + 2\sqrt{\Lambda_1 \Lambda_2} \geq 0.$$

Next we show $a \pm b + c \geq 0$ for all $\phi_1 \geq 0, \phi_2 \geq 0$.

Proposition 6. $a - b + c \geq 0$ for all $\phi_1 \geq 0, \phi_2 \geq 0$ if and only if

(1) $\Delta \leq 0, \Lambda_6\sqrt{\Lambda_2} + \Lambda_7\sqrt{\Lambda_1} < 0$;

(2) $\Lambda_6 \leq 0, \Lambda_7 \leq 0, \Lambda_3 + \Lambda_4 + \Lambda_5 + 2\sqrt{\Lambda_1 \Lambda_2} \geq 0$;

(3) $\Delta \geq 0$,

$$|\Lambda_6\sqrt{\Lambda_2} - \Lambda_7\sqrt{\Lambda_1}| \leq 2\sqrt{\Lambda_1 \Lambda_2 (\Lambda_3 + \Lambda_4 + \Lambda_5)} + 2\Lambda_1 \Lambda_2 \sqrt{\Lambda_1 \Lambda_2} \text{ and}$$

$$(i) -2\sqrt{\Lambda_1 \Lambda_2} \leq \Lambda_3 + \Lambda_4 + \Lambda_5 \leq 6\sqrt{\Lambda_1 \Lambda_2},$$

$$(ii) \Lambda_3 + \Lambda_4 + \Lambda_5 > 6\sqrt{\Lambda_1 \Lambda_2} \text{ and}$$

$$\Lambda_6\sqrt{\Lambda_2} + \Lambda_7\sqrt{\Lambda_1} \leq 2\sqrt{\Lambda_1 \Lambda_2 (\Lambda_3 + \Lambda_4 + \Lambda_5)} - 2\Lambda_1 \Lambda_2 \sqrt{\Lambda_1 \Lambda_2},$$

where $\Delta = 4(12\Lambda_1\Lambda_2 - 12\Lambda_6\Lambda_7 + (\Lambda_3 + \Lambda_4 + \Lambda_5)^2)^3 - (72\Lambda_1\Lambda_2(\Lambda_3 + \Lambda_4 + \Lambda_5) + 36\Lambda_6\Lambda_7(\Lambda_3 + \Lambda_4 + \Lambda_5) - 2(\Lambda_3 + \Lambda_4 + \Lambda_5)^3 - 108\Lambda_1\Lambda_7^2 - 108\Lambda_6^2\Lambda_2)^2$.

Proof. From the equations (8) and (9), it follows that

$$\begin{aligned} a - b + c &= \Lambda_1\phi_1^4 + \Lambda_2\phi_2^4 + (\Lambda_3 + \Lambda_4 + \Lambda_5)\phi_1^2\phi_2^2 \\ &\quad - 2\Lambda_6\phi_1^3\phi_2 - 2\Lambda_7\phi_2^3\phi_1 \\ &= f(\phi_1, \phi_2) = a_0\phi_1^4 + a_1\phi_1^3\phi_2 + a_2\phi_1^2\phi_2^2 + a_3\phi_1\phi_2^3 + a_4\phi_2^4, \end{aligned}$$

where $a_0 = \Lambda_1, a_1 = -2\Lambda_6, a_2 = \Lambda_3 + \Lambda_4 + \Lambda_5, a_3 = -2\Lambda_7, a_4 = \Lambda_2$. Then an application of the co-positivity of quartic form (4) (Lemma 1) yields that the sufficient and necessary conditions of the inequality $a - b + c \geq 0$ is

- (1) $\Delta \leq 0, -2\Lambda_6\sqrt{\Lambda_2} - 2\Lambda_7\sqrt{\Lambda_2} > 0;$
- (2) $-2\Lambda_6 \geq 0, -2\Lambda_7 \geq 0, \Lambda_3 + \Lambda_4 + \Lambda_5 + 2\sqrt{\Lambda_1\Lambda_2} \geq 0;$
- (3) $\Delta \geq 0,$
 $|2\Lambda_6\sqrt{\Lambda_2} - 2\Lambda_7\sqrt{\Lambda_1}| \leq 4\sqrt{\Lambda_1\Lambda_2(\Lambda_3 + \Lambda_4 + \Lambda_5)} + 2\Lambda_1\Lambda_2\sqrt{\Lambda_1\Lambda_2}$ and
 - (i) $-2\sqrt{\Lambda_1\Lambda_2} \leq \Lambda_3 + \Lambda_4 + \Lambda_5 \leq 6\sqrt{\Lambda_1\Lambda_2},$
 - (ii) $\Lambda_3 + \Lambda_4 + \Lambda_5 > 6\sqrt{\Lambda_1\Lambda_2}$ and
 $-2\Lambda_6\sqrt{\Lambda_2} - 2\Lambda_7\sqrt{\Lambda_1} \geq -4\sqrt{\Lambda_1\Lambda_2(\Lambda_3 + \Lambda_4 + \Lambda_5)} - 2\Lambda_1\Lambda_2\sqrt{\Lambda_1\Lambda_2}.$

The required conclusions follow. \square

Similarly, the following conclusion is easy to be showed.

Proposition 7. $a + b + c \geq 0$ for all $\phi_1 \geq 0, \phi_2 \geq 0$ if and only if

- (1) $\Delta \leq 0, \Lambda_6\sqrt{\Lambda_2} + \Lambda_7\sqrt{\Lambda_1} > 0;$
- (2) $\Lambda_6 \geq 0, \Lambda_7 \geq 0, \Lambda_3 + \Lambda_4 + \Lambda_5 + 2\sqrt{\Lambda_1\Lambda_2} \geq 0;$
- (3) $\Delta \geq 0,$
 $|\Lambda_6\sqrt{\Lambda_2} - \Lambda_7\sqrt{\Lambda_1}| \leq 2\sqrt{\Lambda_1\Lambda_2(\Lambda_3 + \Lambda_4 + \Lambda_5)} + 2\Lambda_1\Lambda_2\sqrt{\Lambda_1\Lambda_2}$ and
 - (i) $-2\sqrt{\Lambda_1\Lambda_2} \leq \Lambda_3 + \Lambda_4 + \Lambda_5 \leq 6\sqrt{\Lambda_1\Lambda_2},$
 - (ii) $\Lambda_3 + \Lambda_4 + \Lambda_5 > 6\sqrt{\Lambda_1\Lambda_2}$ and
 $\Lambda_6\sqrt{\Lambda_2} + \Lambda_7\sqrt{\Lambda_1} \geq -2\sqrt{\Lambda_1\Lambda_2(\Lambda_3 + \Lambda_4 + \Lambda_5)} - 2\Lambda_1\Lambda_2\sqrt{\Lambda_1\Lambda_2}.$

Combining Propositions 4 (1) with Propositions 5 and 7, the following results are easy to obtain.

Proposition 8. $A + B + C \geq 0$ for all $\phi_1 \geq 0, \phi_2 \geq 0$ if and only if $\Lambda_3 + 2\sqrt{\Lambda_1\Lambda_2} \geq 0$ and

- (1) $\Lambda_6 = \Lambda_7 = 0, \Lambda_4 + \Lambda_5 \geq 0;$
- (2) $\Delta \leq 0, \Lambda_6\sqrt{\Lambda_2} + \Lambda_7\sqrt{\Lambda_1} > 0;$
- (3) $\Lambda_6 \geq 0, \Lambda_7 \geq 0, \Lambda_3 + \Lambda_4 + \Lambda_5 + 2\sqrt{\Lambda_1\Lambda_2} \geq 0;$
- (4) $\Delta \geq 0,$
 $|\Lambda_6\sqrt{\Lambda_2} - \Lambda_7\sqrt{\Lambda_1}| \leq 2\sqrt{\Lambda_1\Lambda_2(\Lambda_3 + \Lambda_4 + \Lambda_5)} + 2\Lambda_1\Lambda_2\sqrt{\Lambda_1\Lambda_2},$
 - (i) $-2\sqrt{\Lambda_1\Lambda_2} \leq \Lambda_3 + \Lambda_4 + \Lambda_5 \leq 6\sqrt{\Lambda_1\Lambda_2},$
 - (ii) $\Lambda_3 + \Lambda_4 + \Lambda_5 > 6\sqrt{\Lambda_1\Lambda_2}$ and
 $\Lambda_6\sqrt{\Lambda_2} + \Lambda_7\sqrt{\Lambda_1} \geq -2\sqrt{\Lambda_1\Lambda_2(\Lambda_3 + \Lambda_4 + \Lambda_5)} - 2\Lambda_1\Lambda_2\sqrt{\Lambda_1\Lambda_2}.$

Simultaneously applying Proposition 5 and Proposition 6 to Propositions 4 (2), the following results are easy to obtain.

Proposition 9. $A - B + C \geq 0$ for all $\phi_1 \geq 0$, $\phi_2 \geq 0$ if and only if $\Lambda_3 + 2\sqrt{\Lambda_1\Lambda_2} \geq 0$ and

- (1) $\Lambda_6 = \Lambda_7 = 0, \Lambda_4 + \Lambda_5 \geq 0$;
- (2) $\Delta \leq 0, \Lambda_6\sqrt{\Lambda_2} + \Lambda_7\sqrt{\Lambda_1} < 0$;
- (3) $\Lambda_6 \leq 0, \Lambda_7 \leq 0, \Lambda_3 + \Lambda_4 + \Lambda_5 + 2\sqrt{\Lambda_1\Lambda_2} \geq 0$;
- (4) $\Delta \geq 0$,

$$|\Lambda_6\sqrt{\Lambda_2} - \Lambda_7\sqrt{\Lambda_1}| \leq 2\sqrt{\Lambda_1\Lambda_2(\Lambda_3 + \Lambda_4 + \Lambda_5) + 2\Lambda_1\Lambda_2\sqrt{\Lambda_1\Lambda_2}},$$

$$(i) -2\sqrt{\Lambda_1\Lambda_2} \leq \Lambda_3 + \Lambda_4 + \Lambda_5 \leq 6\sqrt{\Lambda_1\Lambda_2},$$

$$(ii) \Lambda_3 + \Lambda_4 + \Lambda_5 > 6\sqrt{\Lambda_1\Lambda_2} \text{ and}$$

$$\Lambda_6\sqrt{\Lambda_2} + \Lambda_7\sqrt{\Lambda_1} \leq 2\sqrt{\Lambda_1\Lambda_2(\Lambda_3 + \Lambda_4 + \Lambda_5) - 2\Lambda_1\Lambda_2\sqrt{\Lambda_1\Lambda_2}}.$$

By combining Propositions 2 with Propositions 3, 8 and 9, we easily showed our main result, Theorem 1.

The proof of Theorem 1. It follows from Propositions 2 and 3 that $V_H(\Phi_1, \Phi_2) \geq 0$ if and only if

$$\begin{aligned} \Lambda_6 = \Lambda_7 = 0, \Lambda_5 \geq 0, \Lambda_3 + \Lambda_4 - \Lambda_5 + 2\sqrt{\Lambda_1\Lambda_2} &\geq 0, \\ \Lambda_3 + 2\sqrt{\Lambda_1\Lambda_2} &\geq 0; \\ A - B + C &\geq 0; \\ A + B + C &\geq 0. \end{aligned} \tag{11}$$

Propositions 8 (3) and 9 (3) mean that

$$\Lambda_6 = \Lambda_7 = 0, \Lambda_3 + 2\sqrt{\Lambda_1\Lambda_2} \geq 0, \Lambda_3 + \Lambda_4 + \Lambda_5 + 2\sqrt{\Lambda_1\Lambda_2} \geq 0;$$

Propositions 8 (1) and 9 (1) imply that

$$\Lambda_6 = \Lambda_7 = 0, \Lambda_3 + 2\sqrt{\Lambda_1\Lambda_2} \geq 0, \Lambda_3 + \Lambda_4 + \Lambda_5 + 2\sqrt{\Lambda_1\Lambda_2} \geq 0,$$

The above several inequalities together is equivalent to

$$\Lambda_6 = \Lambda_7 = 0, \Lambda_3 + 2\sqrt{\Lambda_1\Lambda_2} \geq 0, \Lambda_3 + \Lambda_4 - |\Lambda_5| + 2\sqrt{\Lambda_1\Lambda_2} \geq 0.$$

This obtain (1) of Theorem 1.

The two inqualities $A - B + C \geq 0$ and $A + B + C \geq 0$ imply that there is contradiction between Proposition 8 (2) and Proposition 9 (2), and so, it can not hold.

By Propositions 8 (4) and 9 (4), (2) of Theorem 1 follows easily. \square

5 Conclusions

In summary, we prove the analytical sufficient and necessary conditions of the vacuum stability conditions of 2HDM potential with explicit CP conservation. At the same time, for a 4th-order 2-dimensional symmetric tensor $\mathcal{A}(\rho, x) = (a_{ijkl})$ (3) with two parameters $\rho \in [0, 1]$ and $x \in [-1, 1]$, the co-positivity is proved. Which first gives a argument methods to study the tensor with two parameters.

6 Some remarks

When $\lambda_6 = \lambda_7 = 0$, Theorem 1 coincides with the corresponding conclusions of Refs. [3, 9, 11] and other references not cited here.

Recently, Bahl et al. [14] gave a stronger sufficient conditions (Eqs. (43) and (44)) of the vacuum stability of CP conserving 2HDM potential with the constraint $\Lambda_6\sqrt{\Lambda_2} = \Lambda_7\sqrt{\Lambda_1}$ using Ulrich and Watson's result (Eq.(30)) [30]. In this paper, we use the optimizing version (Lemma 1) of Ulrich and Watson's result to yield the analytic expression of the vacuum stability. For the more details about Lemma 1 see Refs. [16, 29].

In terms of the eigenvalues of a 4×4 matrix, Ivanov [6] gave the sufficient and necessary conditions of the vacuum stability of CP conserving 2HDM potential, but not analytic. Our conclusions is analytic in this paper.

Our work also first applies the successive removal parameter method to study the properties of a symmetric tensor with multi-parameters, which deserves further study and perfectness.

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Data Availability Statement

This manuscript has no associated data or the data will not be deposited. [Authors' comment: This is a theoretical study and there are no external data associated with the manuscript.]

Appendix

The proof of Propositions 3. From the equation (5), it follows that

$$\begin{aligned} B - 2A &= 2(\Lambda_6\phi_1^2 + \Lambda_7\phi_2^2 - 2\Lambda_5\rho\phi_1\phi_2)\rho\phi_1\phi_2, \\ B + 2A &= 2(\Lambda_6\phi_1^2 + \Lambda_7\phi_2^2 + 2\Lambda_5\rho\phi_1\phi_2)\rho\phi_1\phi_2, \end{aligned}$$

and hence,

$$\begin{aligned} B - 2A \leq 0 &\Leftrightarrow -\Lambda_6\phi_1^2 - \Lambda_7\phi_2^2 + 2\Lambda_5\rho\phi_1\phi_2 \geq 0, \\ B + 2A \geq 0 &\Leftrightarrow \Lambda_6\phi_1^2 + \Lambda_7\phi_2^2 + 2\Lambda_5\rho\phi_1\phi_2 \geq 0. \end{aligned}$$

Thus, $B - 2A \leq 0$ is equivalent to the co-positivity of a matrix $M = (m_{ij})$ with its entries $m_{11} = -\Lambda_6, m_{22} = -\Lambda_7, m_{12} = m_{21} = \Lambda_5\rho$. From the co-positivity of 2×2 matrix, it follows that

$$B - 2A \leq 0 \Leftrightarrow \Lambda_6 \leq 0, \Lambda_7 \leq 0 \text{ and } \Lambda_5\rho + \sqrt{\Lambda_6\Lambda_7} \geq 0.$$

Similarly, we also have

$$B + 2A \geq 0 \Leftrightarrow \Lambda_6 \geq 0, \Lambda_7 \geq 0 \text{ and } \Lambda_5\rho + \sqrt{\Lambda_6\Lambda_7} \geq 0.$$

Thus, the inequalities $-2A \leq B \leq 2A$ imply that $\Lambda_6 = 0, \Lambda_7 = 0, \Lambda_5 \geq 0$, i.e., $B = 0, A \geq 0$, and so, the inequality $4AC - B^2 \geq 0$ means that

$$C = \Lambda_1\phi_1^4 + \Lambda_2\phi_2^4 + [\Lambda_3 + (\Lambda_4 - \Lambda_5)\rho^2]\phi_1^2\phi_2^2 \geq 0.$$

which is equivalent to the co-positivity of 2×2 matrix $M = (m_{ij})$ with its entries,

$$m_{11} = \Lambda_1, \quad m_{22} = \Lambda_2, \quad m_{12} = m_{21} = \frac{1}{2}[\Lambda_3 + (\Lambda_4 - \Lambda_5)\rho^2].$$

That is, $\Lambda_3 + (\Lambda_4 - \Lambda_5)\rho^2 + 2\sqrt{\Lambda_1\Lambda_2} \geq 0$. Therefore, this is equivalent to

$$\begin{aligned} \Lambda_3 + \Lambda_4 - \Lambda_5 + 2\sqrt{\Lambda_1\Lambda_2} &\geq 0, \\ \Lambda_3 + 2\sqrt{\Lambda_1\Lambda_2} &\geq 0. \end{aligned}$$

The required conclusion follows. \square

The proof of Propositions 5. (1) It is obvious that

$$c = \Lambda_1\phi_1^4 + \Lambda_2\phi_2^4 + \Lambda_3\phi_1^2\phi_2^2 \geq 0,$$

is equivalent to

$$\Lambda_1 \geq 0, \Lambda_2 \geq 0, \Lambda_3 + 2\sqrt{\Lambda_1\Lambda_2} \geq 0.$$

(2) It follows from the equations (8) that

$$b + 2a = 2(\Lambda_6\phi_1^2 + \Lambda_7\phi_2^2 + (\Lambda_4 + \Lambda_5)\phi_1\phi_2)\phi_1\phi_2,$$

$$2a - b = 2(-\Lambda_6\phi_1^2 - \Lambda_7\phi_2^2 + (\Lambda_4 + \Lambda_5)\phi_1\phi_2)\phi_1\phi_2,$$

and hence,

$$\begin{aligned} b + 2a \geq 0 &\Leftrightarrow \Lambda_6\phi_1^2 + \Lambda_7\phi_2^2 + (\Lambda_4 + \Lambda_5)\phi_1\phi_2 \geq 0, \\ b \leq 0 &\Leftrightarrow \Lambda_6\phi_1^2 + \Lambda_7\phi_2^2 \leq 0. \end{aligned}$$

So, we have

$$\begin{aligned} b + 2a \geq 0 &\Leftrightarrow \Lambda_6 \geq 0, \Lambda_7 \geq 0, (\Lambda_4 + \Lambda_5) + 2\sqrt{\Lambda_6\Lambda_7} \geq 0, \\ b \leq 0 &\Leftrightarrow \Lambda_6 \leq 0, \Lambda_7 \leq 0, \end{aligned}$$

and hence, $\Lambda_4 + \Lambda_5 \geq 0$, $\Lambda_6 = 0$, $\Lambda_7 = 0$, i.e., $b = 0$. Clearly, if $a \neq 0$, then

$$\frac{4ac - b^2}{4a} \geq 0 \Leftrightarrow c \geq 0.$$

Similarly, (3) is also showed easily. \square

References

- [1] T. Lee, A Theory of Spontaneous T Violation, Phys. Rev. D 8, 1226(1973); CP nonconservation and spontaneous symmetry breaking, Physics Reports, 9(2) 143-177(1974)
- [2] S. Weinberg, Gauge Theory of CP Nonconservation, Phys. Rev. Lett. 37, 657 (1976)
- [3] N. G. Deshpande, E. Ma, Pattern of symmetry breaking with two Higgs doublets, Phys. Rev. D 18, 2574-2576(1978)
- [4] A. Barroso, P.M. Ferreira, I.P. Ivanov, R. Santos, Metastability bounds on the two Higgs doublet model, J. High Energy Phys. 06, 045(2013)
- [5] G. Chauhan, Vacuum stability and symmetry breaking in left-right symmetric model. J. High Energ. Phys. 2019, 137 (2019).

- [6] I. P. Ivanov, Minkowski space structure of the Higgs potential in the two-Higgs-doublet model Phys. Rev. D 75, 035001; Erratum Phys. Rev. D 76, 039902 (2007)
- [7] K. Kannike, Vacuum stability of a general scalar potential of a few fields. Eur. Phys. J. C, 76, 324(2016); Erratum. Eur. Phys. J. C, 78, 355(2018)
- [8] K. Kannike, Vacuum stability conditions from copositivity criteria. Eur. Phys. J. C, 72, 2093(2012)
- [9] K. G. Klimenko, On Necessary and Sufficient Conditions for Some Higgs Potentials to Be Bounded From Below, Theor. Math. Phys. 62, 58-65 (1985)
- [10] G.C. Branco, P.M. Ferreira, L. Lavoura, M.N. Rebelo, M. Sher, J.P. Silva, Theory and phenomenology of two-Higgs-doublet models, Phys. Rep. 516, 1 (2012)
- [11] I.F. Ginzburg, M. Krawczyk, Symmetries of two Higgs doublet model and CP violation, Phys. Rev. D 72, 115013(2005)
- [12] M. Maniatisa, A. von Manteuffelb, O. Nachtmannc, F. Nagel, Stability and symmetry breaking in the general two-Higgs-doublet model, Eur. Phys. J. C 48, 805-823 (2006)
- [13] I. P. Ivanov and João P. Silva, Tree-level metastability bounds for the most general two Higgs doublet model. Phys. Rev. D 92, 055017(2015)
- [14] H. Bahl, M. Carena, N. M. Coyle, A. Ireland, Carlos E.M. Wagner, New Tools for Dissecting the General 2HDM, arXiv:2210.00024 (2022)
- [15] J. Liu, Y. Song, Copositivity for 3rd order symmetric tensors and applications. Bull. Malays. Math. Sci. Soc. 45(1), 133-152(2022)
- [16] Y. Song, X. Li, Copositivity for a class of fourth order symmetric tensors given by scalar dark matter, J. Optim Theory Appl. 195, 334–346 (2022)
- [17] Y. Song, L. Qi, Analytical expressions of copositivity for fourth-order symmetric tensors. Anal. Appl., 19(5), 779-800(2021)
- [18] Y. Song, L. Qi, A necessary and sufficient condition of positive definiteness for 4th order symmetric tensors defined in particle physics. arXiv: 2011.11262 (2020)
- [19] Y. Song, Positive definiteness for 4th order symmetric tensors and applications. Anal. Math. Phys. 11, 10(2021)
- [20] L.E. Andersson, G. Chang, T. Elfving, Criteria for copositive matrices using simplices and barycentric coordinates. Linear Algebra Appl. 5, 9-30(1995)

- [21] K.P. Hadeler, On copositive matrices. *Linear Algebra Appl.* 49, 79-89(1983)
- [22] E. Nadler, Nonnegativity of bivariate quadratic functions on a triangle. *Comput. Aided Geom. D.* 9, 195-205(1992)
- [23] H. Ishimori, T. Kobayashi, H. Ohki, Y. Shimizu, H. Okada, M. Tanimoto, Non-Abelian Discrete Symmetries in Particle Physics, *Prog. Theor. Phys. Suppl.* 183, 1-163 (2010)
- [24] L. Qi, Eigenvalues of a real supersymmetric tensor, *J. Symbolic Comput.*, 40(6) 1302-1324(2005)
- [25] L. Qi, Symmetric Nonnegative Tensors and Copositive Tensors, *Linear Algebra Appl.*, 439, 228-238(2013)
- [26] Y. Song, L. Qi, Necessary and sufficient conditions for copositive tensors, *Linear Multilinear A* 63(1), 120-131(2015).
- [27] L. Qi, H. Chen, Y. Chen, *Tensor Eigenvalues and Their Applications*, Springer Singapore, 2018.
- [28] L. Qi, Z. Luo, *Tensor Analysis: Spectral Theory and Special Tensors*, SIAM, Philadelphia 2017.
- [29] L. Qi, Y. Song, X. Zhang, Positivity Conditions for Cubic, Quartic and Quintic Polynomials, *J. Nonlinear Convex Anal.* 23(2), 191-213(2022)
- [30] G. Ulrich, L.T. Watson, Positivity conditions for quartic polynomials. *SIAM J. Sci. Comput.* 15, 528-544(1994)